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Adjoint Variable Solutions via an Auxiliary Optimization Problem

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The problem addressed is to obtain an initial guess for the adjoint variables along a trajectory suitable for use in an indirect optimal control method. The proposed method assumes that good estimates of the control and state histories are available. The adjoint estimates are obtained through an auxiliary unconstrained optimization problem of low dimension. The goal of this optimization problem is to find the adjoint values at nodes along the trajectory that cause the control values from the application of the minimum principle to approach the estimated control values as closely as possible in a manner consistent with the adjoint differential equations and transversality conditions. Experience has shown that the method produces adjoint variable estimates sufficiently accurate to cause convergence of a multiple-shooting indirect method.

Nomenclature

$\mathcal{H}$ = Hamiltonian
$\boldsymbol{u}$ = p-dimensional control vector
$\boldsymbol{x}$ = n-dimensional vector of state variables
$\boldsymbol{\lambda}$ = n-dimensional vector of adjoint variables
$\boldsymbol{\Psi}$ = m-dimensional vector of constraints at terminal time, $m < n$

Introduction

NONLINEAR optimal control solutions of high-fidelity aerospace dynamics problems are generally easy to formulate, but more often than not they are difficult to impossible to solve. One class of methods for obtaining solutions is the indirect method, namely to solve the mixed two-point boundary value problem resulting from applying the maximum principle (or minimum principle in the formulation of Bryson and Ho), and corresponding transversality conditions, to the system of state and associated adjoint differential equations.[1,2] Among this class of solution techniques, the multiple-shooting algorithm is perhaps the most successful.[3,4] This algorithm breaks the integration span of the independent variable into numerous smaller spans in order to mitigate error growth caused by the inherent instability of nearly all state and adjoint systems. The unknown states and/or adjoints at the initial point, and possibly at interior nodes, are iterated upon, using an equation solver such as that based on the quasi-Newton method, to find the values that cause the state and adjoint values at the final point to match the prescribed conditions.

The difficulty with any indirect method is that guesses for the unknown values of the state and adjoint variables that are in the neighborhood of the optimal solution must be made. This is especially critical for the multiple-shooting method, which requires guesses of the state and adjoint variables at a number of points along the trajectory. Although a reasonable first guess to the optimal state solution is often possible based on experience with the dynamic system and technical intuition, this is generally not so with the adjoint variables. Even though these guesses for the values of the adjoint variables only need to be in the "valley" of the solution to be able to converge, even this level of accuracy is generally impossible for the adjoint variables, since their solution is far from intuitive.[5,6]

It is with this in mind that an auxiliary optimization problem is presented in this paper that provides approximate solutions to the adjoint variables. This auxiliary optimization problem is to find the adjoint variable values at given node points along a vehicle's flight path using a finite difference scheme for the adjoint differential equations, which minimize the difference between the value of the control variables as derived from the maximum principle and the value of the control variables provided by an approximate solution method. In other words, the total proposed optimal control solution technique is a three-step method. The first step is to obtain an approximate optimal solution via any method that produces state and control variable estimates at nodes along the trajectory. The second step is to apply the auxiliary optimization technique to provide a good guess for the adjoint history, which in turn provides control solutions that closely match the approximate optimal solutions. The third step is to provide the approximate state and adjoint solutions from the first two steps as an initial guess to a multiple-shooting method.

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General Indirect Method

The general optimal control problem, to which the auxiliary optimization problem is applied, is given by the following classical equations and conditions:

$$\min_{u} \mathcal{J}[u]$$

where

$$\mathcal{J}[u] = \phi[\boldsymbol{x}(t_f), t_f] + \int_{t_0}^{t_f} L(\boldsymbol{x}, \boldsymbol{u}, t)\,\mathrm{d}t \tag{1}$$

subject to

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}, t), \qquad \boldsymbol{\Psi}[\boldsymbol{x}(t_f), t_f] = 0, \qquad \boldsymbol{x}(t_0) = x_0$$

The set of $2n$ differential equations to be solved are the well-known Euler–Lagrange equations given as

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}, t) \tag{2}$$

and

$$\dot{\boldsymbol{\lambda}} = -\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}}\right)^T = -\left(\frac{\partial f}{\partial \boldsymbol{x}}\right)^T \boldsymbol{\lambda} - \left(\frac{\partial L}{\partial \boldsymbol{x}}\right)^T \tag{3}$$

where

$$\mathcal{H} = \boldsymbol{\lambda}^T f + L \tag{4}$$

The minimum principle necessary condition is that the optimal control minimizes $\mathcal{H}$ on the optimal trajectory:

$$\boldsymbol{u}^* = \arg\min_{u} \mathcal{H}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{u}, t) \tag{5}$$

The boundary conditions for the mixed two-point boundary value problem are given by the $2n$ transversality conditions with

$$\lambda(t_f) = \left[\frac{\partial \phi}{\partial \boldsymbol{x}} + \nu^T \frac{\partial \Psi}{\partial \boldsymbol{x}}\right]^T\Bigg|_{t=t_f} \tag{6}$$

$$\mathcal{H}(t_f) = -\left[\frac{\partial \phi}{\partial t} + \nu^T \frac{\partial \Psi}{\partial t}\right]\Bigg|_{t=t_f} \tag{7}$$

$$\lambda(t_0) = \text{free parameters to be determined} \tag{8}$$

and by $(n+m)$ terminal state constraints

$$\Psi[\boldsymbol{x}(t_f), t_f] = 0 \tag{9}$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{10}$$

The indirect method, in this case, has reduced the optimal control problem to the task of finding the parameters $\lambda(t_0)$, ν and t_f such that the conditions (6), (7), and (9) are satisfied. The multiple-shooting method is one of the more successful methods for solving boundary value problems of this type.

State and Control Variable Estimation

The first step in obtaining the optimal control solution to the above equations is to obtain an approximate solution for the state and control variables. This can be done by any method, but a parameter optimization scheme applied to a discretized control history is particularly well suited for obtaining the desired solution.[7] The approach detailed in Ref. 7 has been found to work quite well and produces excellent estimates of the state and control variables for many problems. The estimates of the state and control variables at points along the trajectory are useful in the second step, the auxiliary optimization problem, which is solved to obtain the adjoint estimates. The state and adjoint estimates are then used during the final step, the solution to the complete optimal control problem, using a multiple-shooting method. The estimates of the control and state variables along the trajectory are denoted by $\tilde{\boldsymbol{u}}_i$ and $\tilde{\boldsymbol{x}}_i$, respectively, where $i = 1, \ldots, w$, and w is the number of trajectory nodes.

Auxiliary Optimization Problem

When solving optimal control problems by employing indirect methods, obtaining a plausible initial guess for the adjoint variables proves quite difficult. Even when a good estimate of the adjoint variables is available at one point in the trajectory, integrating the adjoint differential equations will not necessarily provide adjoint estimates along the trajectory that are sufficiently accurate to cause convergence of the multiple-shooting algorithm. For example, in the case of a parameterized control solution, the Lagrange multipliers of the terminal constraints provide an estimate of the adjoint variables at the final time[8]; when the adjoint variables are integrated backward, however, the control values resulting from applying the minimum principle are generally markedly different from the estimated control. The result is that the multiple-shooting algorithm does not generally converge.

The key to successful estimation of the adjoint variables is to anchor the control values from the form of the minimum principle (that is, as functions of the adjoint variables) to the estimated control values, in a manner consistent with the adjoint differential equations and transversality conditions. This is the fundamental concept behind the auxiliary optimization problem. The adjoint differential equations are handled with a finite difference scheme that uses the estimated state and control values from the first step to compute the adjoint derivatives at each node, and the freedom in the transversality conditions are iterated upon until the minimum principle control matches the estimated control as closely as possible, in a sense made explicit below.

To employ the finite difference method, the adjoint variable differential equations (3) are rewritten as

$$\dot{\lambda} = [A(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}, t)]\lambda - \frac{\partial L}{\partial \boldsymbol{x}}\bigg|_{\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}, t} \tag{11}$$

where

$$A = -\left(\frac{\partial f}{\partial \boldsymbol{x}}\right)^T \tag{12}$$

Equation (11) can then be approximated as

$$[\boldsymbol{I}]\boldsymbol{\lambda}_{i+1} - [\boldsymbol{I} + A_i(\tilde{\boldsymbol{x}}_i, \tilde{\boldsymbol{u}}_i, t_i)\,\Delta t_i]\hat{\boldsymbol{\lambda}}_i = -\Delta t_i \frac{\partial L}{\partial \boldsymbol{x}}\bigg|_{\tilde{\boldsymbol{x}}_i, \tilde{\boldsymbol{u}}_i, t_i} \qquad i = 1, \ldots, w-1 \tag{13}$$

The scalar interval between nodes, not required to be uniform throughout, is given by

$$\Delta t_i = t_{i+1} - t_i \tag{14}$$

Expanding Eq. (13) in matrix notation,

$$\begin{bmatrix} \alpha_1 & \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \alpha_2 & \boldsymbol{I} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \alpha_{w-1} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \\ \vdots \\ \boldsymbol{\lambda}_{w-1} \\ \boldsymbol{\lambda}_w \end{Bmatrix} = \begin{Bmatrix} -\Delta t_1 \frac{\partial L}{\partial \boldsymbol{x}}\Big|_1 \\ -\Delta t_2 \frac{\partial L}{\partial \boldsymbol{x}}\Big|_2 \\ \vdots \\ -\Delta t_{w-1} \frac{\partial L}{\partial \boldsymbol{x}}\Big|_{w-1} \\ \boldsymbol{\lambda}(t_f) \end{Bmatrix} \tag{15}$$

shows the linear system that needs to be solved for the $\boldsymbol{\lambda}_i$. In the above equation, $\boldsymbol{0}$ and $\boldsymbol{I}$ denote, respectively, the null and identity $n \times n$ sub-matrices, while each $\boldsymbol{\lambda}_i$ denotes an $n \times 1$ subvector. The $n \times n$ submatrix term α_i in the above matrix expansion is given by

$$\alpha_i = -(\boldsymbol{I} + \boldsymbol{A}_i \;\; \Delta t_i)$$

This first-order forward-difference scheme has been found to yield solutions sufficiently accurate to be used as initial guesses for the multiple-shooting algorithm. Of course, higher order methods could be substituted for Eq. (15) if greater accuracy is necessary.

The final adjoint variable $\boldsymbol{\lambda}(t_f)$ on the right side of Eq. (15) is computed by [compare with Eq. (6)]

$$\boldsymbol{\lambda}(t_f) = \left[\frac{\partial \phi}{\partial \boldsymbol{x}} + z^T \frac{\partial \Psi}{\partial \boldsymbol{x}}\right]^T \tag{16}$$

where the partials on the right side of Eq. (16) are evaluated using the estimated final state and time. The vector $z \in \Re^{n-m}$ is to be chosen such that the values of the adjoint variables at the nodes from the solution to the linear system equation (15) yield control values from the minimum principle such that the sum of the squared differences between these control values and the $\tilde{\boldsymbol{u}}_i$ is minimized. Specifically, the minimum principle yields a control function of the form

$$\boldsymbol{u}^*(t) = q(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, t)$$

where the asterisk denotes quantities on the optimal trajectory. Given the specific form of $q(\cdot)$ for a particular problem, the control at the node points can be estimated as

$$\hat{\boldsymbol{u}}_i(z) = q(\tilde{\boldsymbol{x}}_i, \hat{\boldsymbol{\lambda}}_i(z), t_i) \qquad i = 1, \ldots, w \tag{17}$$

The circumflex denotes quantities that vary as z varies. It is important to note that the original estimates $\tilde{\boldsymbol{u}}_i$ are used on the right side of Eq. (17) through Eqs. (13) and (15) in order to compute the $\hat{\boldsymbol{u}}_i$ from the form of the minimum principle. The auxiliary optimization problem is to find the optimal z^* that satisfies

$$\min_z \sum_{i=1}^{w} \|\hat{\boldsymbol{u}}_i(z) - \tilde{\boldsymbol{u}}_i\|^2 \tag{18}$$

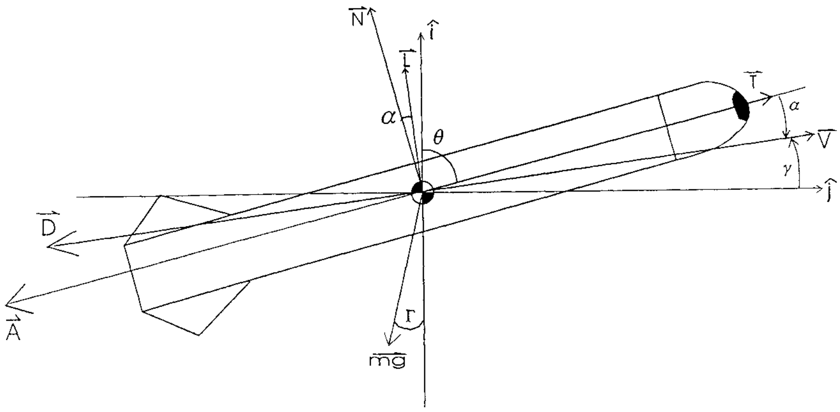

Fig. 1 Forces acting on the missile.

The solution to the auxiliary optimization problem determines, through Eqs. (13) and (18), the desired set of estimates for the adjoint variables, $\tilde{\boldsymbol{\lambda}}_i$.

With the now complete set of state and adjoint variable estimates, solution to the differential equations (2) and (3), using the control defined by Eq. (5) and the boundary conditions given in Eqs. (6–10), can be obtained using a multiple-shooting algorithm.

Illustrative Example

Equations of Motion

In order to illustrate the auxiliary optimization problem to obtain the approximate adjoints and to illustrate the full-solution approach from start to finish, the problem of finding the maximum specific kinetic energy to a point in space in a fixed amount of time for a multiple-stage missile is worked in detail. This is a particularly significant problem for solid propellant missiles, where the burn time and thrust levels of the missile are fixed and there is a requirement to achieve the highest terminal velocity to a point inside the engagement envelope. The missile dynamics are modeled as given in Ref. 7, and are repeated here for completeness, as vertical plane motion over a spherical nonrotating Earth:

$$\dot{x} = u \tag{19a}$$

$$\dot{y} = v \tag{19b}$$

$$\dot{u} = -\frac{\mu}{r^3}x + \frac{T}{m}\cos\theta - \frac{A}{m}\cos\theta + \frac{N}{m}\sin\theta \tag{19c}$$

$$\dot{v} = -\frac{\mu}{r^3}y + \frac{T}{m}\sin\theta - \frac{A}{m}\sin\theta - \frac{N}{m}\cos\theta \tag{19d}$$

where μ, T, A, and N are the gravitational parameter, thrust force, axial force, and normal force, respectively. The control is the angle θ, which defines the missile longitudinal axis as measured from the vertical axis at missile launch. The following definitions are used in the terms of the above equations:

$$r^2 = x^2 + y^2 \tag{20a}$$

$$V^2 = u^2 + v^2 \tag{20b}$$

$$A = \tfrac{1}{2}\rho V^2 S C_A \tag{20c}$$

$$N = \tfrac{1}{2}\rho V^2 S C_{N_\alpha}\alpha \tag{20d}$$

$$\alpha = \tfrac{1}{2}\pi - \theta - \tan^{-1}(u/v) \tag{20e}$$

$$M = V/a \tag{20f}$$

$$C_A = C_A(M, a) \tag{20g}$$

where the density ρ and speed of sound a are altitude dependent and the normal force curve slope C_{N_α} is a constant over each stage. Also stage dependent are the missile mass (which is time dependent within each stage) m, the reference area S, and the axial force coefficient C_A. The above equations and definitions apply to Fig. 1.

Optimal Control and Boundary Conditions

The performance objective is to maximize the terminal specific kinetic energy for a specified fixed time of flight. There are no path contributions to the objective function or the constraints, but there are terminal state constraints. Therefore, the specific forms of Eqs. (1) and (9) are

$$L = 0$$
$$\phi[\boldsymbol{x}(t_f)] = -\tfrac{1}{2}(u^2 + v^2)|_{t=t_f} \tag{21}$$

and

$$\Psi[\boldsymbol{x}(t_f)] = \begin{Bmatrix} x - x_f \\ y - y_f \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{22}$$

With the state equations for our application given by Eqs. (19), the state variable boundary conditions on the trajectory are given as

$$\boldsymbol{x}(t_0) = \{x_0 \quad y_0 \quad u_0 \quad v_0\}^T \tag{23a}$$

$$\boldsymbol{x}(t_f) = \{x_f \quad y_f \quad \mu_1 \quad \mu_2\}^T \tag{23b}$$

where μ_1 and μ_2 are free parameters.

It should be noted that the Hamiltonian is explicitly time dependent through its reliance upon the mass of the missile and is therefore not a constant while the missile is thrusting. Also, since mass is dropped between stages, the Hamiltonian is discontinuous at the staging point. However, after thrusting terminates, the Hamiltonian is constant at some nonzero value.

The boundary conditions for the adjoint variables are given by the transversality conditions with

$$\boldsymbol{\lambda}(t_0) = \{\mu_3 \quad \mu_4 \quad \mu_5 \quad \mu_6\}^T \tag{24a}$$

$$\boldsymbol{\lambda}(t_f) = \{0 \quad 0 \quad -u \quad -v\}^T|_{t=t_f} + \left[\begin{Bmatrix} \nu_1 \\ \nu_2 \end{Bmatrix}^T \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Bmatrix}\right]^T\Bigg|_{t=t_f} \tag{24b}$$

The equation for the optimal control θ is given by

$$\tan\theta = -\frac{\lambda_v[(T-A) + N/\alpha] + \lambda_u N}{\lambda_v N - \lambda_u[(T-A) + N/\alpha]} \tag{25}$$

In order to determine proper quadrant resolution for the expression in Eq. (25), the second-order condition $\partial^2\mathcal{H}/\partial\boldsymbol{u}^2 > 0$ is applied. The results of the condition show that the minus sign in front of the above expression should be applied to the numerator when determining the correct quadrant for the optimal control.

It should be noted that in Eq. (25) the normal force N is dependent on α and hence θ, so an analytic solution is not readily obtainable. Therefore, during the third step the optimal control θ is calculated numerically using a Newton method with Eq. (25). However, to be consistent during the auxiliary optimization problem, Eq. (25) is a direct expression for the estimate of the control, $\hat{u}$, because $\tilde{x}$ and $\tilde{u}$ are used to evaluate all terms on the right side of Eq. (25).

The axial force coefficient C_A and the partial derivative of the axial force coefficient with respect to Mach number, $\partial C_A/\partial M$, are evaluated analytically using the axial force coefficient expression given in Ref. 7. The partial derivatives $\partial a/\partial h$ and $\partial \rho/\partial h$ are evaluated numerically using a subroutine that models the 1962 U.S. Standard Atmosphere Density Data.

Auxiliary Optimization Problem

For our four-state system, given the set of state and control estimates at the nodes, $\tilde{x}$ and $\tilde{u}$, the auxiliary optimization problem is to determine the values of the parameters $z = [z_1 \quad z_2]$ that minimize the expression

$$\sum_{i=1}^{w} \|\hat{u}_i(t_i, \tilde{x}_i, \lambda_i) - \tilde{u}_i\|^2$$

subject to the solution of the system of $4w$ equations (w being the number of irregularly spaced nodes along the trajectory):

$$[I]\hat{\lambda}_{i+1} - [I + A_i(\tilde{x}_i, \tilde{u}_i, t_i)\,\Delta t_i]\hat{\lambda}_i = 0 \qquad i = 1, \ldots, w-1$$

and

$$\lambda_w = \{z_1 \quad z_2 \quad -u \quad -v\}^T$$

where each 4×4 submatrix A_i is evaluated with the state and control variable estimates, $\tilde{x}$ and $\tilde{u}$, using the right-hand sides of Eq. (12). The estimate of the control at each node, $\hat{u}_i$, is evaluated using Eq. (25) and is evaluated in the manner described previously. Solving the minimization problem yields the set of $4w$ values for the estimate of the adjoint variables $\tilde{\lambda}_i$ needed for the multiple-shooting algorithm.

At this point, the multiple-shooting algorithm is executed to obtain the full optimal control solution. The details of the multiple-shooting algorithm are given in Ref. 4. It has been the authors' experience that with state estimates provided via the parameter optimization and adjoint estimates provided by the auxiliary optimization, solutions were easy to obtain. In most instances, the solutions were obtained within 5–10 iterations using the multiple-shooting algorithm.

Numerical Results

For the numerical example, the multistage missile configuration is modeled as a two-stage missile of moderate performance capability. It has a maximum burnout velocity of 13,350 ft/s and a launch weight of 2000 lb. Table 1 summarizes the physical parameters of the missile.

As an example, the optimal control solution to reach 35 km down range and 40 km altitude in 30 s of flight time is determined. The missile is to reach a given terminal point with a coasting segment after burnout. At burnout, the controlling force changes from the thrust to the aerodynamic normal force. The resulting optimal trajectory profile is given in Fig. 2. Note that in order to reach the specified terminal position in the fixed time of flight, the missile must perform an energy management maneuver by first flying away from the target point and then pitching back over toward it, in a manner that preserves maximum final velocity.

Table 1 Missile parameters

Parameter	First stage	Second stage
Propellant weight, lb	900.0	577.5
Inert weight, lb	300.0	122.5
Specific impulse, s	267.0	270.0
Burn time, s	8.0	17.0
Reference diameter, in.	20.0	15.0
Normal force curve slope	2.8	2.8

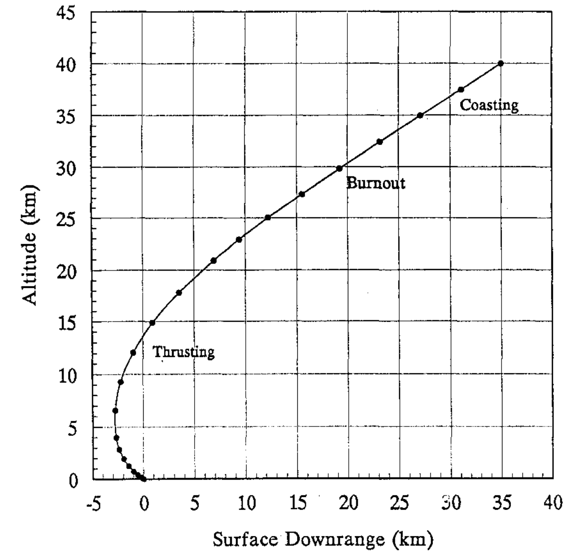

Fig. 2 Optimal trajectory path to maximize terminal velocity.

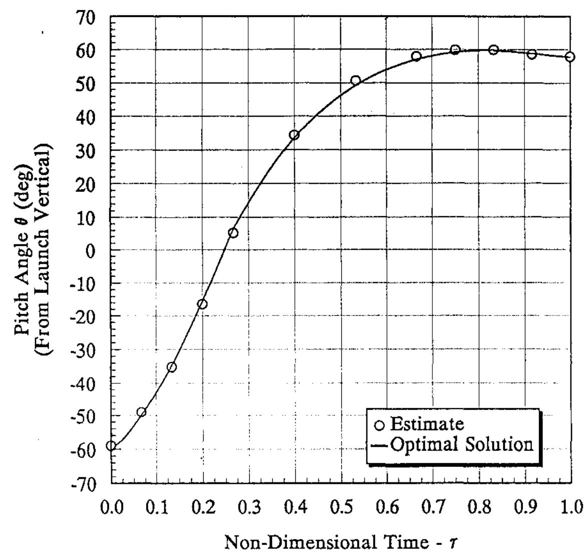

Fig. 3 Estimate and optimal solution for the control.

The control history to achieve the above optimal trajectory is shown in Fig. 3. The 10 open circles or nodes indicate the estimate of the control from the approximate solution. The spacing of the nodes along the trajectory history is arbitrary with the only restriction being that a node must fall at the staging point. Linear interpolation between the nodes provides continuity of the control variable θ. The estimate of the control variable at the nodes is obtained by varying their value until the terminal constraints are satisfied using the subroutine NLPQL developed by Schittkowski.[9] The solid line is the optimal control obtained by the multiple-shooting algorithm.

The optimal control shown in Fig. 3 is calculated from the adjoint variables obtained by the auxiliary optimization and multiple-shooting steps. For our example, Fig. 4 shows the converged optimal solution for the position adjoint variables λ_x and λ_y as a function of the nondimensional time $\tau = t/t_f$ as well as the initial estimates provided to the multiple-shooting algorithm. For illustrative purposes, to show the instability of the adjoint variable differential equations, the equations were integrated forward from node to node, resetting the integration initial value to the value of the estimate at each succeeding node. At staging times, optimal control theory states that the adjoint variables will be continuous, but the Hamiltonian may be discontinuous.[10] The discontinuities that do appear in the figures are due to the prior node's initial estimated

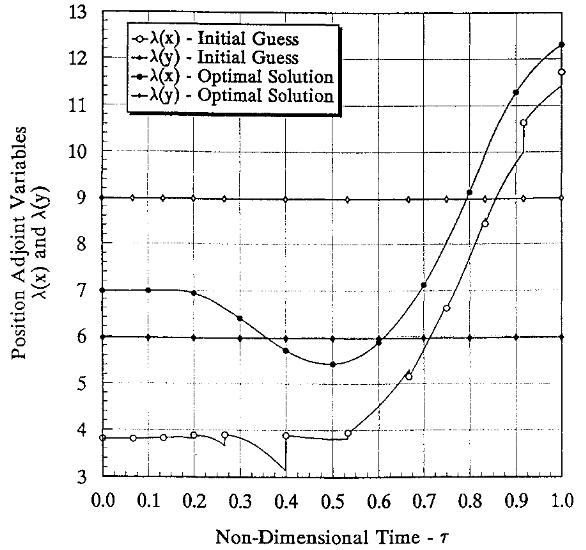

Fig. 4 Estimate and optimal solution for the position adjoints λ_x and λ_y.

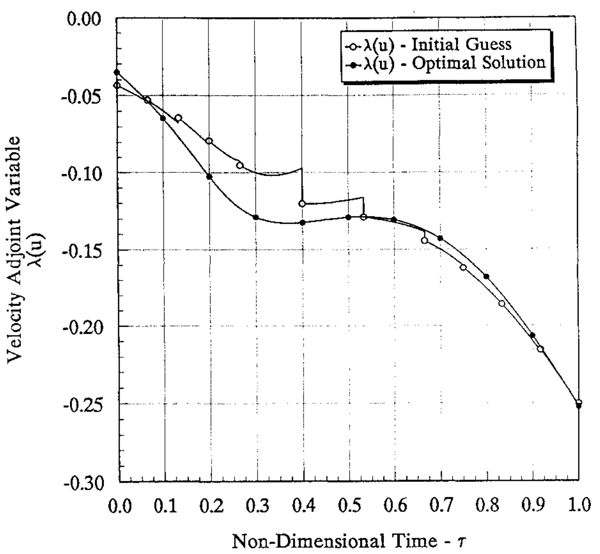

Fig. 5 Estimate and optimal solution for the velocity adjoint λ_u.

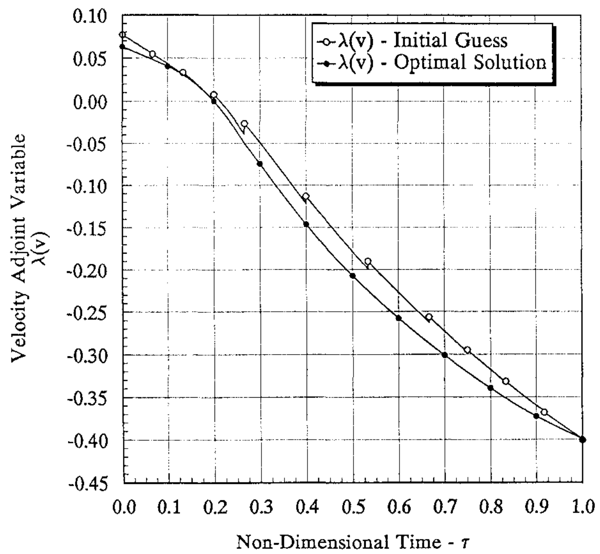

Fig. 6 Estimate and optimal solution for the velocity adjoint λ_v.

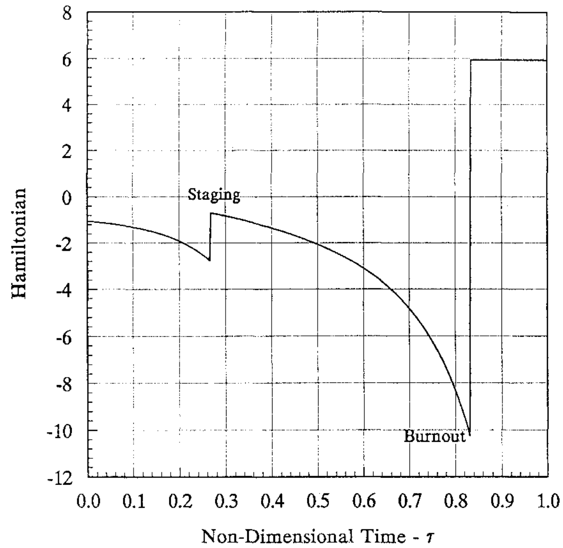

Fig. 7 Hamiltonian vs time.

value being integrated forward and then reset at the next node; these discontinuities illustrate what is meant by multiple shooting. For a proper solution we shoot from node to node and, as required by optimal control theory, iterate until continuity is achieved at each node.

Figures 5 and 6 show the converged optimal values of the velocity adjoint variables λ_u and λ_v, respectively, as functions of nondimensional time. Again, for illustrative purposes, the optimal solution is shown along with the estimates integrated from node to node. Figure 7 shows the Hamiltonian history for this example. Note that after missile burnout the mass of the vehicle is constant, causing the Hamiltonian to be a nonzero constant value as shown.

Applicability and Extensions

The solution technique presented herein is particularly well suited to problems where the control varies continuously throughout the trajectory. It is not yet known how well the presented method handles problems with bang-bang controls, path constraints, and singular arcs. Bang-bang controls often have switching structures that facilitate the adjoint estimation process, such as in primer vector theory of optimal orbit transfer.[11,12] However, the method presented herein should also be applicable as long as care is taken in the approximation of the state and control. For problems with path constraints and singular arcs, all indications are that the proposed method will work well, as long as times to enter and exit path constraints and singular arcs are fairly well approximated. Since the method of Seywald provides excellent estimates for these times,[5] it could possibly be used in conjunction with the auxiliary optimization method to provide good estimates to a multiple-shooting method.

Finally, the method presented herein was derived assuming the initial state is completely specified. This led to an unconstrained optimization problem to approximate the adjoints. If there is some freedom in the initial states, then an initial transversality condition analogous to Eq. (6) (less the cost function term) must be applied to the initial values of adjoints. Through Eq. (15), this in turn puts a linear constraint on z in the auxiliary optimization problem. With this modification, the method presented herein can be extended to handle this more general case.

Conclusions

A method for obtaining an estimate of the adjoint variables along the trajectory when a near optimal control and the corresponding state history are given has been presented. The combined state and adjoint estimates can then be used as initial guesses for a multiple-shooting method. These adjoint estimates are obtained by solving an auxiliary optimization problem that forces the control values from the form of the minimum principle to match the estimated control values at the node points along the trajectory as closely as possible, in a manner consistent with the adjoint differential equations and transversality conditions.

For the numerical example presented, the present method requires two parameter optimization solutions, one for the state variables and the other for the adjoint variables, each with a low number of variables and very low number of constraints. This is significant because the ability to find solutions to parameter optimization problems is inversely related to the number of states and the number of constraints. For this example, the size of the parameter optimization state space is equal to the number of nodes with only two constraints. The auxiliary optimization problem state space is two dimensional with no constraints.

From the numerical example, it is seen that the obtained estimates for the adjoint variables capture the form of the corresponding final solutions obtained with the multiple-shooting method. Thus the method provides a reasonable initial guess for the adjoint variables, which is so important in the field of optimization.

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